Modeling non-convex configuration space using linear complementarity problems

Binh Nguyen[1] and Jeff Trinkle[2]

***Abstract*— In this paper, we proposed a new physical simulation method that can model non-convex configuration space. The new method employs a novel contact model that take into account geometry information of objects. It can also be shown that it reduces the work for collision detection routines.**

I. Introduction

Physical simulation of rigid bodies with Coulomb friction has come a long way since Lötstedt first developedcomplementarity problems formulations [1], [2]. Till now, there are various methods that are capable of handling complicated tasks from video games and virtual reality [3], [4], to graphics and haptic application [5], to robotics, machine design and virtual prototyping [6], [7], [8], [9], [10], [11]. However, they all share a common practice: the contact model between a vertex and a face is used to prevent penetration. This simple contact model is easy to use but cannot handle cases where the shape of the free space in the neighborhood of the contact point is non-convex like the case in figure 1. Moreover, it requires complicated collision detection that is often not trivial to implement. In fact, getting correct colliding information and penetration vectors in collision detection for this simple contact is an open problem. In this paper, we discuss a new extended contact model that can accurately model locally non-convex contact problems for rigid body systems. We also show that it requires simpler collision detection than the traditional model.

A. Background

The goal of a time-stepping method is to produce estimates of a dynamic system states at a discrete set of times. Given the state at the current time, the task is to formulate a time-stepping subproblem whose solution yields the state at the next time in the set. Typical time-stepping methods for multi rigid bodies systems with unilateral contacts accomplish this by cycling between two main functions: one that computes distances between geometric features of the bodies, and one that takes this list, formulates the dynamics subproblem, solves it, and updates the system state. This process is repeated until the final time is reached.

[1]Binh Nguyen is PhD student at Computer Science Department, Rensselaer Polytechnic Institute, Troy, NY, USA nguyeb2@cs.rpi.edu

[2]Jeff Trinkle is Professor of Computer Science Department, Rensselaer Polytechnic Institute, Troy, NY, USA trink@cs.rpi.edu

The most important geometric information in most general methods is the set of geometric features (vertices and faces or edges and edges) that are in contact (distance = 0), nearly so (distance is positive, but small) or in penetration (distance is negative). We refer to the list of such feature pairs as the *active* (contact) *set*. This set is needed to formulate the dynamics subproblem so that it can prevent inter-penetration of rigid bodies. The active set is normally obtained from a collision detection software package.

Current time-stepping methods can be classified into two main types: correction methods [3], [12], [9] report only pairs that are in penetration or touching and prevention methods [6], [7], [13] that also report pairs that could collide in the next time step. In correction methods, constraints are formed to stop the current penetrations from getting worse. An additional correction steps can be used to reduce the depth of penetration or eliminate it all together. Even though correction methods can employ error-correction step, they have a major drawback: penetration is unavoidable due to its *wait-and-correct* behavior and to numerical errors during simulation. In general, penetration should be avoided during simulation where possible, not just because it is physically incorrect but also because there are cases where geometric information is not sufficient to recover from a penetration. In correction methods, penetration depth scales with simulation time step and object speed. It makes correction methods sensitive to those runtime factors. This dependence increases the complexity of collision detection for correction methods. It has to treat high speed objects, bullets for example, in a different manner than normal objects to avoid 'tunneling' effect i.e objects passing through others in one time step. Reducing the simulation time step to anticipate high-speed objects is not a good option because object's speed can vary widely while a fixed time step is crucial for realtime simulation. Smaller time steps also increase the computing power required to simulate a fixed duration.

Prevention methods [6], [7], [13] have less penetration during simulation, because they anticipate penetrating pairs before collisions. These methods also have built-in correction to eliminate of penetrations caused by linearization and numerical errors. This type of method is also less sensitive to the value of the time step and object speed because they can prevent the 'tunneling' effect by cautiously activating contact pairs that include the high speed objects. Prevention methods also require different active sets than correction ones. They need collision detection to include not only penetrating and in-contact pairs but also potentially colliding pairs. This difference explains the lack of prevention methods in popular physics engines as all current collision detection routines are only designed to work with corrective methods. The only implementation of this type of method known to us is our simulation package, dVC [14], which uses a simple heuristic that compares geometric distances between possible pairs and a velocity-sensitive constant to choose the active set. It works well in general, but the collision detection is also very complicated especially in three-dimensional cases.

B. Simple contact model and locally non-convex configuration space

The common contact model in current methods is the simple one between a point and a face (or edge in planar case). An active constraint associated with this simple contact model keeps the distance ψ_{in} between its point and face from becoming negative. Usually for the ith active contact, the constraint has the form:

$$0 \leq \lambda_{in} \perp \psi_{in}(\mathbf{q}, t) \geq 0 \tag{1}$$

where ψ_{in} is a signed distance function or *gap function* for the ith contact with the property $\psi_{in}(\mathbf{q}, t) > 0$ implies separation, $\psi_{in}(\mathbf{q}, t) = 0$ implies touching, and $\psi_{in}(\mathbf{q}, t) < 0$ for interpenetration. Note that in general, there is no closed-form expression for $\psi_{in}(\mathbf{q}, t)$ so usually approximation values are used instead. λ_{in} is the force or impulse needed to prevent ψ_{in} from becoming negative.

Compactly, we can write the non-penetration constraint for all contacts as

$$0 \leq \boldsymbol{\lambda}_{\rm n} \perp \boldsymbol{\psi}_{\rm n}(\mathbf{q}, t) \geq 0 \tag{2}$$

where $\boldsymbol{\psi}_{\rm n}$ and $\boldsymbol{\lambda}_{\rm n}$ are the concatenated vectors of all the signed distance functions and normal forces (impulses) respectively.

Equation (2) over-constrains the system at those contacts where the local configuration space is non-convex. It can be illustrated in two simple cases below.

In figure 1, if we write the constraints in form (2) then the result must satisfy : $\psi_{1\rm n} \geq 0$ and $\psi_{2\rm n} \geq 0$ which means in the next time step, the particle P cannot leave cone (I) while physically, it should be allowed to pass through cone (II) and (IV). In the case of figure 2, formulation (2) is infeasible as there is no point in space that has non negative distances with all lines that support the active edges (the ones that intersect with the circle interior in the figure).

In conclusion, current methods not only cannot simulate the particular cases shown in figures 1 and 2 but also rely on complicated collision detection routines to find the active set. It is not trivial to implement collision detection algorithms needed for penetration prevention. In fact, the current development of continuous collision detection suggests that only geometric information is not enough to find active set. Even so, one can always find the cases where collision detection fails to identify the right active set with this simple contact model.

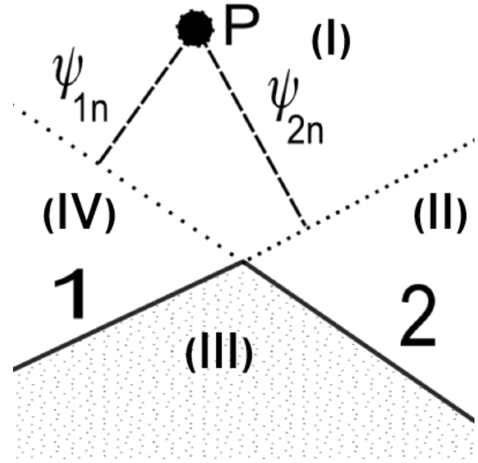


Fig. 1: Locally non-convex C-space

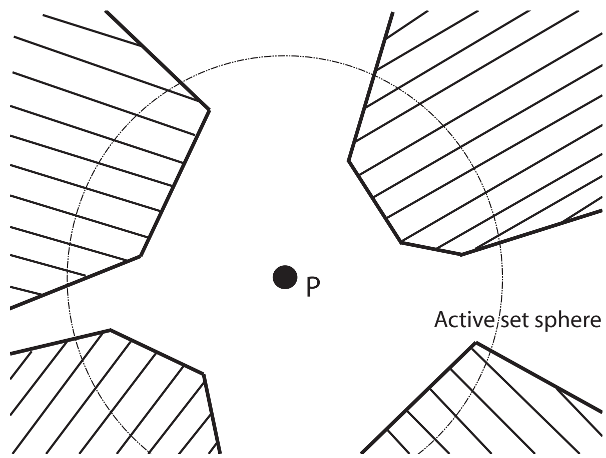


Fig. 2: A more complicated scene resembling a robot moving through a room

C. Previous works and summary of results and contributions

Kevin Egan [15] proposed a way to handle non-convex configurations but his method requires a trial-and-error parameter tuning process to approximate the non-convex region. Also, it is not easy to relate the formulation used in [15] to physical laws. The method proposed in this paper can be shown to closely follow the underlying physics model and also accurately captures the non-convexity.

Recently, Glocker et al [11] proposed a class of methods

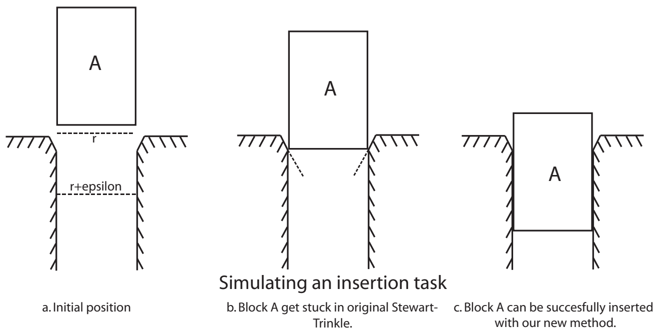


Fig. 3: A test case where robot need to insert a rectangular block into a closely fitted hole

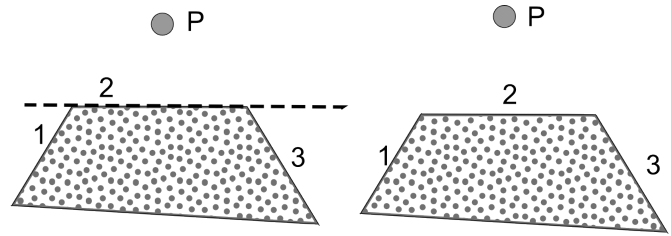


Fig. 4: Left: current contact model (P,2) virtually extends edge 2, thus prevent P from entering region below the dotted line. Right: new contact model (P,[1,2,3]) accurately capture the shape of obstacle and P can move freely outside dotted region.

that do not use complementarity conditions but rather use mathematically equivalent prox functions to model contact. These methods can also be classified as prevention or correction because they use the same simple contact model and rely on the same collision detection to identify the active set. Note that it is easy to switch between prox function and complementarity forms. The prox function form loses the physical interpretation of the complementarity one.

This paper extends the current simple contact model to include not only the geometric features but also information about the relations between them. The main advantages of the new contact model proposed in this work are two fold: it solves the problem that all current simple contact models have with locally non-convex configuration spaces and it reduces the dependency of the simulation results on the internal details of collision detection.

II. Modeling non-convex configuration space

A. New contact model

This new contact model begins with the assertion that we need at most one impulse to prevent penetration between a vertex and a convex shape. Thus, the new contact between two bodies is defined by:

- A vertex in the first body.
- A convex portion of the second body.

The physical meaning of current contact model described in [3], [6], [9] is that when activated, it prevents the vertex from penetrating the half space that contains the contact face. The new contact model extends the idea by replacing that one contact face with a convex shape (defined by a list of faces) and physically prevents the contact vertex from penetrating the shape interior.

It is worth noting that we can choose any convex subset of the second geometric body or the whole body itself if it is convex. If the second body geometry is convex, any convex hulls of a subset of its vertices belong to the body geometry also. Thus, it is easy to generate new contact information for a convex body: a simple heuristic picks all facets with Euclidean distance to contact vertex smaller than a certain threshold to include in the new contact should work well. If the shape of second body is not convex, this new contact model can still accurately model the object by decomposing the shape into a list of convex shapes, then attach one contact to each of them when needed.

It is easy to see that the new contact model not only solves the problems with contacts that have locally non-convex configuration spaces, it also gives the users more freedom in forming the active sets. Usually, it is not obvious how to determine which facet will make contact with a vertex in next time step to form the active set. This new contact model gives the collision detection the flexibility to handle this uncertainty better: it can be conservative by picking a big convex shape when it is hard to guess the correct set or even just one facet if it is obvious (see figure 5). Normally, for current methods, the only way to tackle uncertainty in set selection problem is to reduce time step or to use a continuous collision detection algorithm which is expensive and could lead to exhaustive exponential search. In reality, all current common physics engines rely on their collision detection ability to pick the right active set. Picking the wrong one would result in unrealistic behaviors. Extra contacts in the active set lead to unwanted obstacles during simulation while missing contacts lead to deep penetrations. The new contact model reduces collision detection complexity because it does not need the unique contact facet corresponds with the contact vertex but a list of possible facets that contain it. With this new contact model, collision detection is well defined and easier to implement. Actually, one can heuristically pick the set of facets to form the contact then calculate the geometric distances between the vertex and all the facets in the list then feed that information to the dynamic step. The correct active facet and contact forces will be obtained along with solving the dynamics subproblem.

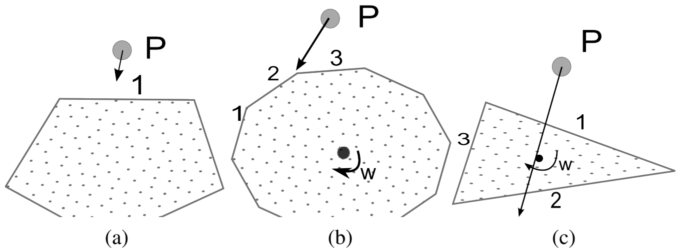


Fig. 5: There are cases where user can use current information to choose the set of edges to minimize the chance of penetration.(a): the vertex moves relatively slow and is close to the obstacle so only edge 1 is needed.(b):the vertex moves fast and the object below is rotating so edge 1,2 and 3 need to be included.(c):Speeds of the vertex and object's spinning are very high, so we may need to consider adding ALL edges.

B. Mathematical implementations

In order to handle the new contact model correctly, we need to keep the vertex from penetrating the convex shape in the next time step. This task can be separated into two main sub-tasks: accurately model the non-convex free space correspond to the contact shape geometry, find the correct facet in the shape that the vertex collides with, then find the correct normal (force) impulse to prevent penetration. We separate the task into two because they are actually different. In following parts, we will provide mathematical equations for each part,we start with a simple contact (with only two edges or faces) then extend the formulation to a general one.

1) Simple contact: We have shown that for the case of locally non-convex configuration spaces, treating all non-penetration constraints conjunctively will lead to error. For the case shown in figure 1, we should not constraint *both* of $\psi_{1\text{n}}$ and $\psi_{2\text{n}}$ but only one of them to be nonnegative or equivalently:

$$\max(\psi_{1\text{n}}, \psi_{2\text{n}}) \geq 0 \tag{3}$$

We leverage the existing Linear Complementarity framework to encode equation (3).

Lemma 2.1: Given $a, b \in \mathbb{R}$, $b = \max(a, 0) \iff 0 \leq b - a \perp b \geq 0$

Proof:

$\Longrightarrow$: from $b = \max(a, 0)$ need $0 \leq b - a \perp b \geq 0$

- case $a \leq 0$: $\max(a, 0) = 0 \Rightarrow b = 0 \Rightarrow 0 \leq b - a \perp b \geq 0$.
- case $a > 0$:$\max(a, 0) = a \Rightarrow b = a \Rightarrow 0 \leq b - a \perp b \geq 0$.

$\Longleftarrow$: from $0 \leq b - a \perp b \geq 0$ need $b = \max(a, 0)$.

- case $b - a = 0, b \geq 0$: obviously $b = \max(a, 0) = a$
- case $b = 0, b - a \geq 0$: because $a \leq 0$ so $\max(a, 0) = 0 = b$

■

Lemma 2.2: Given $a, b \in \mathbb{R}$, $b = |\min(a, 0)| \iff 0 \leq b + a \perp b \geq 0$

Proof: We call $b = |\min(a, 0)|$ and $0 \leq b + a \perp b \geq 0$ as equation (i) and (ii) respectively.

$\Longrightarrow$: (i) holds, need to prove (ii)

- case $a \leq 0$: $|\min(a, 0)| = -a = b \Rightarrow b + a = 0 \Rightarrow$ (ii) holds.
- case $a > 0$:$|\min(a, 0)| = 0 = b \Rightarrow (a + b)b = 0 \Rightarrow$ (ii) holds.

$\Longleftarrow$: (ii) holds, need to prove (i).

- case $b + a = 0, b \geq 0 \Rightarrow a \leq 0 \Rightarrow |\min(a.0)| = -a = b$
- case $b = 0, b + a > 0 \Rightarrow a > 0 \Rightarrow |\min(a.0)| = 0 = b$

■

Using these two lemmas, we can transform the constraint (3) into linear complementarity conditions as follow:

$$\max(\psi_{1\text{n}}, \psi_{2\text{n}}) = \psi_{2\text{n}} + \max(\psi_{1\text{n}} - \psi_{2\text{n}}, 0) \tag{4}$$

Then define the variable c as follows:

$$c = \max(\psi_{1\text{n}} - \psi_{2\text{n}}, 0) \tag{5}$$

Using lemma 2.1 equation (5) can be written as a linear complementarity condition:

$$0 \leq c - (\psi_{1\text{n}} - \psi_{2\text{n}}) \perp c \geq 0 \tag{6}$$

Then, the correct non-penetration constraint at the contact along edge 1 is:

$$0 \leq c + \psi_{2\text{n}} \perp \lambda_{1\text{n}} \geq 0 \tag{7}$$

This constraint basically means that when the term $c + \psi_{2\text{n}}$, which is equivalent to $\max(\psi_{1\text{n}}, \psi_{2\text{n}})$, becomes negative, the normal force (impulse) $\lambda_{1\text{n}}$ along edge 1 will be positive to prevent the penetration.

Similarly, non-penetration constraint along edge 2 is:

$$0 \leq c + \psi_{2\text{n}} \perp \lambda_{2\text{n}} \geq 0 \tag{8}$$

Here, $\lambda_{1\text{n}}$ and $\lambda_{2\text{n}}$ are normal forces(or impulses) to maintain condition (3). The above non-penetration constraint correctly prevents penetration at the contact but it allows impulses to be generated on both edges, which is not physically correct. There should only be at most one impulse along the edge that the vertex will collide with in the next time step. So we need a constraint that allows no more than one of $\lambda_{1\text{n}}$ and $\lambda_{2\text{n}}$ to be positive.

Lemma 2.3: Given $a, b \in \mathbb{R}$, $a = 0, b \leq 0 \Leftrightarrow (\max(a, b) \geq 0) \wedge (\max(a, b) + |\min(a, 0)| = 0)$

Proof: Because $\max(a, b) \geq 0$ and $|\min(a, 0)| \geq 0$ then the only way to have $\max(a, b) + |\min(a, 0)| = 0$ is both reach equality. From $|\min(a, 0)| = 0$ we can have $a \geq 0$ so to keep $\max(a, b) = 0$ then $a = 0, b \leq 0$. The reverse direction is obvious. ■

Using lemma 2.3, we can formulate a non-penetration constraint along edge 1 as:

$$
\begin{aligned}
0 \leq & c - (\psi_{1\text{n}} - \psi_{2\text{n}}) \perp \quad c \geq 0 \\
0 \leq & \qquad d_1 + \psi_{1\text{n}} \perp \ d_1 \geq 0 \\
0 \leq & \quad c + \psi_{2\text{n}} + d_1 \perp \lambda_{1\text{n}} \geq 0 \\
& \qquad c + \psi_{2\text{n}} \geq \quad 0
\end{aligned} \tag{9}
$$

Equations (9) only allows the normal force (impulse) along edge 1 to be nonnegative when $\psi_{2\text{n}} \leq 0$ and $\psi_{1\text{n}} = 0$. Note that $\psi_{2\text{n}} \leq 0$ and $\psi_{1\text{n}} = 0$ physically means particle P is touching edge 1 in figure 1.

Similarly, a non-penetration constraint along the second edge is:

$$
\begin{aligned}
0 \leq & \quad d_2 + \psi_{2\text{n}} \perp \ d_2 \geq 0 \\
0 \leq & c + \psi_{2\text{n}} + d_2 \perp \lambda_{2\text{n}} \geq 0
\end{aligned} \tag{10}
$$

2) General contact case: In general, the contact has the form $(P, [1, 2, ...n])$. We can extend simple case formulations (9) and (10) as follow:

$$
\begin{aligned}
0 \leq & \qquad c_2 - \psi_{2\text{n}} + \psi_{1\text{n}} \perp \ c_2 \geq 0 \\
0 \leq & \qquad c_3 - \psi_{3\text{n}} + c_2 + \psi_{1\text{n}} \perp \ c_3 \geq 0 \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
0 \leq & c_n - \psi_{nn} + c_{n-1} + \cdots + c_2 + \psi_{1\text{n}} \perp \ c_k \geq 0 \\
0 \leq & \qquad d_1 + \psi_{1\text{n}} \perp \ d_1 \geq 0 \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
0 \leq & \qquad d_n + \psi_{nn} \perp \ d_n \geq 0 \\
0 \leq & \qquad d_1 + c_2 + \cdots + c_n + \psi_{1\text{n}} \perp \lambda_{1\text{n}} \geq 0 \\
& \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
0 \leq & \qquad d_n + c_2 + \cdots + c_n + \psi_{nn} \perp \lambda_{nn} \geq 0 \\
& \qquad c_2 + \cdots + c_n + \psi_{1\text{n}} \geq \quad 0
\end{aligned} \tag{11}
$$

where $c_2, \cdots, c_n$, $d_1, \cdots, d_n$ and e are new variables, $\psi_{1\text{n}}, \psi_{2\text{n}}, \cdots, \psi_{nn}$ are distances between the vertex of this contact and the edges.

A problem arises here, in (11), the last inequality is not in the form of a linear complementarity condition. We will discuss about how to solve this problem in the next section.

In comparison, the same type of constraints based on most of current methods has the form:

$$
0 \leq \psi_{in} \perp \lambda_{in} \geq 0, i = 1 \cdots n \tag{12}
$$

C. Solution method

Here we briefly present two possible solution methods.

1) Optimization method: It is natural to formulate equations (11) as a LPCC (Linear Program with Complementarity Constraints) form [16]:

$$
\begin{aligned}
\textit{minimize} \quad & c' * x + d' * y \\
\textit{subject to} \quad & A * x + B * y \geq f \\
& 0 \leq y \perp q + N * x + M * y \geq 0 \\
& x \geq 0 \\
& x \in \mathbb{R}^n, y \in \mathbb{R}^m, f \in \mathbb{R}^k
\end{aligned}
$$

This is the most robust but slow solution method.

2) Linear Complementarity Problems method: Indeed, we can recast the inequality:

$$
c_2 + \cdots + c_n + \psi_{1\text{n}} \geq 0
$$

as a complementarity condition:

$$
0 \leq c_2 + \cdots + c_n + \psi_{1\text{n}} - \alpha \cdot e \perp e \geq 0
$$

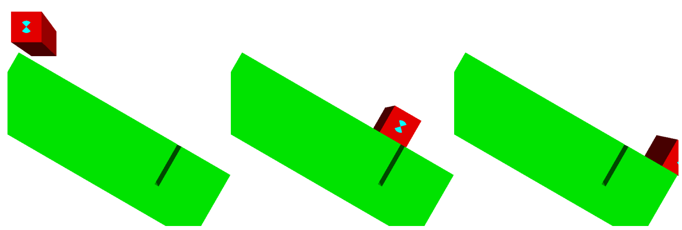
where α is a non-negative number and e is a dummy variable.

III. Numerical results

In this section, we present two examples to compare our new method with the old ones.

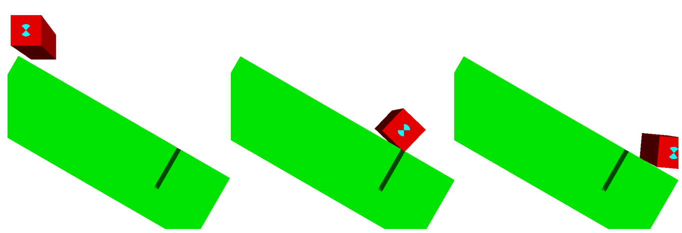
A. Box-ramp

We simulate a box falling then sliding down a cracked ramp. Note that the crack is small so physically, the box should be able to slide to the end of the ramp.



(a) t=0.0000 (b) t=2.7360 (c) t=3.2080

Fig. 6: New method, time step $h = 0.008$



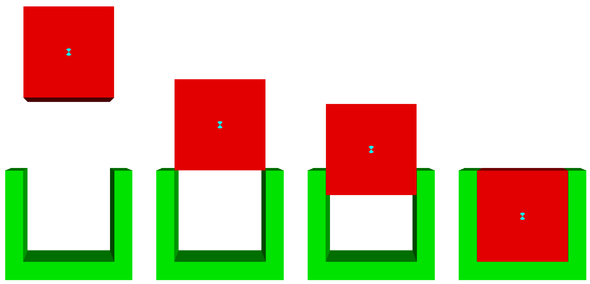
(a) t=0.0000 (b) t=2.7760 (c) t=3.3120

Fig. 7: Old method, time step $h = 0.008$.

With the old contact model, the interaction with the crack caused the block to tumble. This example is motivated by the fact that Computer-Aided Design (CAD) software sometimes produces flawed models.

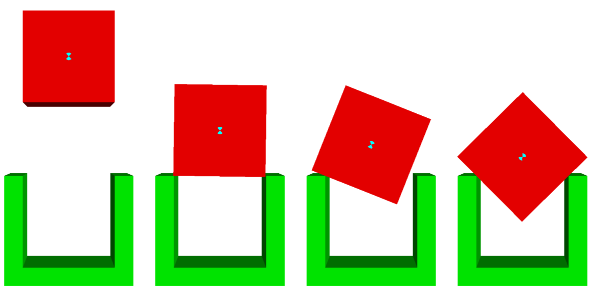
B. Peg in hole

We simulate a box falls under gravity into a hole with a minimal clearance below (box width = 9.99, hole width = 10). Physically, the box should be able to fall through until making contact with the hole's floor.



(a) t=0.0000 (b) t=1.2720 (c) t=1.4720 (d) t=2.0000

Fig. 8: New method, time step = 0.0125.



(a) t=0.0000 (b) t=1.2800 (c) t=1.4000 (d) t=1.6640

Fig. 9: Old method, time step = 0.0125

All examples are solved using PATH [17] using the method mentioned in previous section.

IV. Conclusion and future works

In conclusion, this paper identifies some problems of current contact model that all major physics engines are using. The biggest draw back of simple single vertex-single facet contact model is that it cannot accurately model contact that has locally non-convex configuration space as shown in figures 1,2. This paper presents a new contact model that includes multiple facets that can accurately model such contacts. This new model also reduces the complexity of collision detection routines.

For future works, the authors focus on better solution methods. One possible direction is to use linear complementarity problems over cones because the inequalities in equations (11) form a cone.

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References

[1] P. Lötstedt, "Coulomb friction in two-dimensional rigid body systems," *ZAMM - Zeitschrift fr Angewandte Mathematik und Mechanik*, vol. 61, no. 12, pp. 605–615, 1981.

[2] ——, "Mechanical systems of rigid bodies subject to unilateral constraints," *SIAM Journal on Applied Mathematics*, vol. 42, no. 2, pp. 281–296, 1982. [Online]. Available: http://link.aip.org/link/?SMM/42/281/1

[3] D. Baraff, "Fast contact force computation for nonpenetrating rigid bodies," in *SIGGRAPH '94: Proceedings of the 21st annual conference on Computer graphics and interactive techniques*. New York, NY, USA: ACM, 1994, pp. 23–34.

[4] K. Erleben, "Stable, robust, and versatile multibody dynamics animation," Ph.D. dissertation, University of Copenhagen, 2005.

[5] D. M. Kaufman, S. Sueda, D. L. James, and D. K. Pai, "Staggered projections for frictional contact in multibody systems," *ACM Transactions on Graphics (SIGGRAPH Asia 2008)*, vol. 27, no. 5, pp. 164:1–164:11, 2008.

[6] D. Stewart and J. Trinkle, "An implicit time-stepping scheme for rigid body dynamics with inelastic collisions and coulomb friction," *International Journal of Numerical Methods in Engineering*, vol. 39, pp. 2673–2691, 1996. [Online]. Available: http://www.cs.rpi.edu/~trink/Papers/STijnme96.pdf

[7] J. Trinkle, "Formulations of multibody dynamics as complementarity problems," in *Proceedings, ASME International Design Engineering Technical Conferences*, September 2003, vIB-48342 (no page numbers in CD proceedings). [Online]. Available: http://www.cs.rpi.edu/~trink/Papers/Tidetc03.pdf

[8] N. Chakraborty, S. Berard, S. Akella, and J. Trinkle, "A fully implicit time-stepping method for multibody systems with intermittent contact," in *Robotics: Science and Systems*, June 2007, in press. [Online]. Available: http://www.cs.rpi.edu/~trink/Papers/CBATrss07.pdf

[9] M. Anitescu and F. A. Potra, "Formulating dynamic multi-rigid-body contact problems with friction as solvable linear complementarity problems," *Nonlinear Dynamics*, vol. 14, pp. 231–247, 1997. [Online]. Available: http://dx.doi.org/10.1023/A:1008292328909

[10] M. Anitescu and G. D. Hart, "A constraint-stabilized time-stepping approach for rigid multibody dynamics with joints, contact and friction," *International Journal for Numerical Methods in Engineering*, vol. 60, pp. 2335–2371, 2004. [Online]. Available: http://dx.doi.org/10.1002/nme.1047

[11] R. I. Leine and C. Glocker, "A set-valued force law for spatial coulomb-contensou friction," *European Journal of Mechanics - A/Solids*, vol. 22, no. 2, pp. 193 – 216, 2003. [Online]. Available: http://www.sciencedirect.com/science/article/B6VKW-486GGK5-1/2/e957e6c9de637dbd56589d2a2f4ae6d8

[12] B. Mirtich, "Impulse-based dynamic simulation of rigid body systems," Ph.D. dissertation, University of California, Berkeley, 1996.

[13] S. Berard, J. Trinkle, B. Nguyen, B. Roghani, V. Kumar, and J. Fink, "daVinci code: A multi-model simulation and analysis tool for multi-body systems," in *IEEE International Conference on Robotics and Automation*, April 2007, pp. 2588–2593. [Online]. Available: http://www.cs.rpi.edu/~trink/Papers/BNRTFKicra07.pdf

[14] S. Berard, B. Nguyen, and J. Trinkle, "Davinci code." [Online]. Available: http://www.robotics.cs.rpi.edu/dvc/

[15] J. C. T. Kevin Egan, Stephen Berard, "Modeling nonconvex constraints using linear complementarity," Rensselaer Polytechnic Institute, Tech. Rep., 2003.

[16] J. Hu, J. E. Mitchell, J.-S. Pang, K. P. Bennett, and G. Kunapuli, "On the global solution of linear programs with linear complementarity constraints," *SIAM Journal on Optimization*, vol. 19, no. 1, pp. 445–471, 2008. [Online]. Available: http://link.aip.org/link/?SJE/19/445/1

[17] M. C. Ferris and T. S. Munson, "Path mcp solver." [Online]. Available: http://www.cs.wisc.edu/cpnet/cpnetsoftware/